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Dimensional Crossover in the Large-N Limit

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We consider dimensional crossover for an $O(N)$ Landau–Ginzburg–Wilson model on a d-dimensional film geometry of thickness L in the large-N limit. We calculate the full universal crossover scaling forms for the free energy and the equation of state. We compare the results obtained using "environmentally friendly" renormalization with those found using a direct, non-renormalization-group approach. A set of effective critical exponents are calculated and scaling laws for these exponents are shown to hold exactly, thereby yielding nontrivial relations between the various thermodynamic scaling functions.

KEY WORDS: Renormalization group; finite size scaling; dimensional crossover; scaling functions; effective exponents.

1. INTRODUCTION

Crossover behavior—the interpolation between different effective degrees of freedom as a function of scale—is a ubiquitous phenomenon in nature. An important, experimentally accessible example is that of dimensional crossover, where a film of thickness L provides a significantly different "environment" for fluctuations to that of infinite space. By implementing a renormalization program which is explicitly dependent on the relevant environmental parameters which induce the crossover, such as film thickness, one obtains a globally defined renormalization group (RG) (an "environmentally friendly" RG) whose characteristic functions interpolate between those of different asymptotic fixed points and with which one can calculate crossover scaling functions.(1) Perturbative calculations of the logarithmic derivatives of such scaling functions (effective exponents)(2) gave results which interpolate between known critical exponents(3) and agree well with lattice simulations(4) and high-temperature series approximations.(5)

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Exactly solvable models provide a useful testing ground for ideas on phase transitions and quantum field theory. To date the only truly exactly solvable models, in the sense that all correlation functions are calculable, in any dimension are the Gaussian and the spherical models[6] and variants of these. The former exhibits pathologies, such as the absence of an ordered phase, pathologies absent in the latter. After Stanley[7] established the equivalence, for an infinite lattice, of the partition functions of the $N=\infty$ limit of the $O(N)$ sigma model and the spherical model it was discussed from a field-theoretic point of view by Wilson,[8] whose analysis led to subsequent developments where the model served as the beginning point of a perturbative expansion in $1/N$ (see ref. 9 for a detailed set of reprints on this topic). The original lattice spherical model was solved for both strictly finite geometries and geometries exhibiting a dimensional crossover by Barber and Fisher.[10] Many generalizations have also received attention. For example, a generalization that includes long-range interactions has been studied for fully finite and cylindrical geometries[11] (see Rudnick[12] for a more recent discussion of the model in a purely finite geometry), while Allen and Pathria[13] have recently studied the model's two-point correlation function.

In this paper we wish to test environmentally friendly renormalization in the context of an exact model—the limit $N\to\infty$ of an $O(N)$ Landau–Ginzburg–Wilson model. This model is closely related to the spherical model and the $O(N)$ sigma model, but it contains an additional parameter λ_B, the φ^4 coupling, which away from the critical point governs the crossover to mean-field behavior. It is the field-theoretic formulation that we discuss in the following, our interest being the model in a film geometry.

The free energy scaling functions for the $O(N)$ model in a film geometry with a variety of boundary conditions, but zero external field, have been obtained by Krech and Dietrich[14] in an ε expansion. For $N>1$ their analysis had difficulties due to the presence of Goldstone modes for $T<T_c(\infty)$. Our approach does not suffer from such difficulties and in this paper we obtain the exact scaling function incorporating both external field and temperature in the large-N limit. In the high-temperature regime and for dimensions sufficiently close to, but less than, four the ε expansion results of ref. 14 are in qualitative agreement with the results presented here.

The format of the paper is as follows: in Section 2 we give a brief overview of the large-N limit. We then derive, via a saddlepoint evaluation of the partition function, the scaling functions which incorporate the dimensional crossover and the crossover to mean-field theory for the free energy and equation of state. We analyze in detail the universal scaling limit of these functions and compare with known results for the spherical model. In

Section 3 we analyze the model with the environmentally friendly RG approach and demonstrate how the results of Section 2 are recovered. In Section 4 we calculate a set of effective exponents which are scaling functions that describe the crossovers between d-dimensional, $(d-1)$-dimensional, and mean-field fixed points. In Section 5 we show that the effective exponents satisfy natural analogs of the standard scaling laws, including hyperscaling. Finally we present our conclusions in Section 6.

2. THE LARGE-N LIMIT

The $O(N)$ "microscopic" Landau–Ginzburg–Wilson Hamiltonian for a d-dimensional film geometry of thickness L is given by

$$\mathscr{H}[\varphi] = \int_0^L \int_{-\infty}^{\infty} d^d x \left(\frac{1}{2} \nabla\varphi^a \nabla\varphi^a + \frac{1}{2} r_B(x)\, \varphi^a\varphi^a + \frac{\lambda_B}{4!} (\varphi^a\varphi^a)^2 - H^a(x)\, \varphi^a \right) \tag{2.1}$$

We will consider $3 \leqslant d \leqslant 4$ and assume that all the temperature dependence of the model is contained in the variable r_B. The explicit form of this temperature dependence is not prescribed by the model and requires a separate ansatz.

The partition function Z is obtained by performing the path integral over the order-parameter fields φ^a with the Hamiltonian (2.1). The generator of one-particle irreducible vertex functions $G[\bar{\varphi}]$, where $\bar{\varphi}^a$ is the induced magnetization, is the Legendre transform of $W[H] = -\ln Z$. If the sources H^a and r_B are taken to be homogeneous, then for a translationally invariant system $\bar{\varphi}^a$ is also homogeneous and in the direction of H^a and $G[\bar{\varphi}] = V\Gamma[\bar{\varphi}]$, where V is the volume. It is convenient, however, to retain the general case for the moment. The vertex functions $\Gamma^{(a_1 \cdots a_N)}$ are the objects of primary interest to us, as once these are known, all the correlation functions of the theory can be reconstructed from them.

In general for the $O(N)$ model there are two types of modes: those along the direction picked out by the field, H^a, and those perpendicular to it. If we choose the direction of the field to be given by the unit vector n^a, then using the two projectors $P_l^{ab} = n^a n^b$ and $P_t^{ab} = \delta^{ab} - n^a n^b$, we can decompose a general vertex function into block-diagonal form. We denote a generic vertex function by $\Gamma^{(N)}_{l \cdots lt \cdots t}$, where the number of l and t subscripts indicates whether a longitudinal or a transverse propagator is to be attached to the vertex at the corresponding point. When all subscripts are either l or t we will use a single l or t; for example, $\Gamma^{(N)}_{t\cdots t}$ will be abbreviated $\Gamma^{(N)}_{t}$.

Due to Ward identities it is sufficient to know only the $\Gamma^{(N)}_{t}$, as all the other vertex functions can be reconstructed from these. For the translationally invariant case with homogeneous H the Ward identity $\Gamma^{(1)}_{l}=\Gamma^{(2)}_{t}\bar{\varphi}$ implies the equations of state

$$M^2\bar{\varphi}=H, \qquad \Gamma^{(1)}_{t}=0 \tag{2.2}$$

where $\Gamma^{(2)}_{t}=M^2$. Decomposing $\Gamma^{(ab)}$ yields $\Gamma^{(2)}_{l}$, $\Gamma^{(2)}_{t}$, and $\Gamma^{(2)}_{lt}$ and Ward identities imply

$$\Gamma^{(2)}_{l}=\Gamma^{(2)}_{t}+\frac{\Gamma^{(4)}_{t}}{3}\bar{\varphi}^2 \qquad \text{and} \qquad \Gamma^{(2)}_{lt}=0 \tag{2.3}$$

The large-N limit is taken such that $N\lambda_B$ is held fixed as $N\to\infty$. In this setting it is possible to obtain exact expressions for the vertex functions of the theory. One can do this either by a direct resummation of the Feynman diagrams or via a saddle-point approximation. In the latter approach the introduction of an auxiliary field ψ allows one to reduce the φ^4 interaction to a quadratic interaction term and an integral over the auxiliary field and to perform the now Gaussian φ integral, which yields the effective Hamiltonian $\mathscr{H}_{\rm eff}(\psi, H)$.

In the large-N limit the integral over ψ can be done in a saddle-point approximation, the saddle-point condition $\partial\mathscr{H}_{\rm eff}/\partial M^2=0$ implying[4]

$$\frac{6M^2}{N\lambda_B}=\tau+\tilde{\varphi}^2+\bigcirc \tag{2.4}$$

where

$$M^2=r_B+\psi\sqrt{\frac{N\lambda_B}{3}}, \qquad \tau=\frac{6r_B}{N\lambda_B}, \qquad \tilde{\varphi}=\frac{\bar{\varphi}}{\sqrt{N}}$$

and the origin of r_B has been shifted to cancel any cutoff dependence of $\bigcirc$. The effective Hamiltonian $\mathscr{H}_{\rm eff}$ evaluated at the saddle point then yields the leading large-N behavior of $W[H, M^2]$. A Legendre transform yields $\Gamma[\bar{\varphi}, M^2]+\Gamma_{\rm reg}[r_B, \Lambda]=W+H\bar{\varphi}$, where $\bar{\varphi}=-\partial W/\partial H$ and Γ has been split into a singular part $\Gamma[\bar{\varphi}, M^2]$, which vanishes at the bulk critical

[4] We use the diagrammatic notation of ref. 2, where any cutoff dependence of the diagram has been canceled against part of r_B, and $(-1)^{k-1}/(k-1)!$ times the kth derivative with respect to r_B of a circle with no dots will be represented by a circle with k dots, the dots representing the point at which each derivative acts.

point, and a remaining regular part $\Gamma_{\rm reg}[r_B, \Lambda]$. The function $\Gamma[\bar{\varphi}, M^2]$ determines the singular part of the free energy density to which it is related by $k_B T$.

Using (2.4), we can express the singular part of the free energy per component $\tilde{\Gamma}$ in the form

$$\tilde{\Gamma} = \frac{1}{2}\left\{\frac{3M^d}{g} + \bigcirc - M^2 \bigcirc\right\} \tag{2.5}$$

where $g = N\lambda_B M^{d-4}$, while the regular part is given by

$$\tilde{\Gamma}_{\rm reg}[r_B, \Lambda] = -\frac{3r_B^2}{2N\lambda_B} + \text{cutoff-dependent terms} \tag{2.6}$$

All vertex functions of interest can be obtained by differentiation of (2.5) with the dependence on r_B and $\bar{\varphi}$ specified by (2.4), and given the Ward identities, we need only specify the even transverse vertex functions $\Gamma_t^{(N)}$.

For later convenience, with $H = 0$, we define $\mathscr{E} = \Gamma^{(0,1)}$ and $\mathscr{C} = -\Gamma^{(0,2)}$. When the temperature dependence of r_B is taken to be

$$r_B = \Lambda^2 \frac{T - T_c(\infty)}{T} \tag{2.7}$$

we have $\mathscr{E}$ and $\mathscr{C}$ proportional to the internal energy and specific heat, respectively.[15]

For a d-dimensional film ($d < 4$) with periodic boundary conditions the basic generating diagram is

$$\bigcirc(M, L) = -\frac{\Gamma(-d/2)\, M^d}{(4\pi)^{d/2}} - \frac{2}{(4\pi)^{(d-1)/2}\, \Gamma((d+1)/2)\, L^d} \times \int_0^\infty dq\, \frac{q^d}{\sqrt{q^2+z^2}}\, \frac{1}{\exp(\sqrt{q^2+z^2}) - 1} \tag{2.8}$$

At $M = 0$ we have $\bigcirc(0, L) = -a_d/L^d$, where a_d is the universal number

$$a_d = 2L^d\{\tilde{\Gamma}\,|_{T_c(\infty)} - \tilde{\Gamma}\,|_{T_c(L)}\} = \frac{2\Gamma(d/2)\,\zeta(d)}{\pi^{d/2}} \tag{2.9}$$

In the small-g limit we obtain mean-field results, while the universal scaling form, governed by the limit $g \to \infty$, is

$$\tilde{\Gamma} = \frac{1}{2}(\bigcirc - M^2 \bigcirc) = \frac{\mathscr{G}(d, z) - a_d}{2L^d} \tag{2.10}$$

where $z = ML$, and the scaling function $\mathscr{G}(d, z)$ vanishes at $z = 0$ and is given by

$$\mathscr{G}(d, z) - a_d = \frac{d-2}{d} \sigma_d z^d - \frac{2}{(4\pi)^{(d-1)/2} \Gamma((d+1)/2)} \times \int_0^\infty dq\, q^{d-2} \frac{q^2 + \frac{1}{2}(d-1) z^2}{\sqrt{q^2 + z^2}[\exp(\sqrt{q^2 + z^2}) - 1]} \tag{2.11}$$

For $d = 3$ the result simplifies to

$$\mathscr{G}(3, z) = \frac{z^3}{12\pi} + \frac{1}{2\pi} \int_0^z dy \frac{y^2}{e^y - 1}$$

with $a_3 = \zeta(3)/\pi$.

The tadpole, $\bigcirc(M, L)$, has the useful decomposition

$$\bigcirc(M, L) = -M^{d-2} \mathscr{F}(d, z) + \frac{b_d}{L^{d-2}} \tag{2.12}$$

where

$$b_d = \frac{\Gamma((d-2)/2)\, \zeta(d-2)}{2\pi^{d/2}} \tag{2.13}$$

and $M^{d-2}\mathscr{F}$ vanishes at $M = 0$ with

$$\mathscr{F}(d, z) = \sigma_d - z^{2-d} \left[\frac{2}{(4\pi)^{(d-1)/2} \Gamma((d-1)/2)} \times \int_0^\infty dq \frac{q^{d-2}}{\sqrt{q^2 + z^2}} \frac{1}{\exp(\sqrt{q^2 + z^2}) - 1} - b_d \right] \tag{2.14}$$

and $\sigma_d = -\Gamma((2-d)/2)/(4\pi)^{d/2}$.

The critical temperature is determined by the zero of the right-hand side of (2.4) with $\bar{\varphi} = 0$. Since we have chosen the origin for the parameter r_B to be the bulk critical temperature $T_c(\infty)$, it is convenient to introduce an alternative parameter t_B whose origin is the film critical temperature. In general the two do not coincide, but differ by a shift $\Delta_B(L)$ so that $t_B = r_B + \Delta_B(L)$ and (2.12) implies

$$\Delta_B(L) = \frac{N\lambda_B}{6} \frac{b_d}{L^{d-2}}$$

With the temperature dependence (2.7) we see that

$$\Lambda^2 \frac{T_c(\infty) - T_c(L)}{T_c(L)} = \Delta_B(L) \tag{2.15}$$

The film critical temperature is suppressed relative to the bulk one (since b_d is positive) and scales with the shift exponent $d-2 = 1/\nu(d)$, $\nu(d)$ being the bulk correlation exponent, all of which is in agreement with the lattice results of Barber and Fisher[10] and later calculations of ref. 16. Furthermore, since b_d diverges at $d=3$, we see that for a three-dimensional film the critical temperature $T_c(L)$ is driven to zero and more careful analysis is appropriate.

In the universal ($g \to \infty$) limit $M(\tau, \tilde{\varphi}, L)$ is determined by[5]

$$w = \mathscr{Q}^{-1}(d, z^2) = z^{d-2}\mathscr{F}(d, z), \text{ where } w \equiv \left(\tau + \frac{b_d}{L^{d-2}} + \tilde{\varphi}^2\right) L^{d-2} \tag{2.16}$$

In terms of the basic scaling variables

$$\tilde{x} = \left(\tau + \frac{b_d}{L^{1/\nu(d)}}\right) |\tilde{\varphi}|^{-1/\beta} \qquad \text{and} \qquad \tilde{y} = L\,|\tilde{\varphi}|^{\nu(d)/\beta(d)}$$

with $\nu(d) = 1/(d-2)$ and $\beta(d) = 1/2$ the bulk d-dimensional exponents, $w = (1+\tilde{x})\,\tilde{y}^{1/\nu(d)}$. For the large-$N$ limit, irrespective of whether we consider the universal limit or not, only the combination $\tau + |\tilde{\varphi}|^{1/\beta}$ plays a role. This has significant consequences for the effective exponents to be considered later, since there is a reduction from two to one variable in the scaling functions.

The equation of state is given by

$$\mathscr{Q}(d, w)\, \tilde{\varphi} L^{-1/\beta(d)} = \tilde{H} \tag{2.17}$$

where $\tilde{H} = H/\sqrt{N}$, the asymptotic forms of which are

$$\begin{aligned} \sigma_d^{-\gamma(d)}(1+\tilde{x})^{\gamma(d)}\, \tilde{\varphi}^{\delta(d)} &= \tilde{H} \qquad \text{for} \quad z \to \infty \\ \left(\frac{L}{\sigma_{d'}}\right)^{\gamma(d')} (1+\tilde{x})^{\gamma(d')}\, \tilde{\varphi}^{\delta(d')} &= \tilde{H} \qquad \text{for} \quad z \to 0 \end{aligned} \tag{2.18}$$

[5] More generally, if we do not take the universal limit, we have the more general two-variable scaling form $z^2 = \mathscr{Q}(d, v, w)$, where $v = N\lambda_B L^{d-4}$.

where $\delta(d)=(d+2)/(d-2)$ and $d'=d-1$. Both limiting forms agree with the usual universal form of the equation of state[17] aside from the factors of $\sigma_d^{-\gamma(d)}$ and $(L/\sigma_{d'})^{\gamma(d')}$, which could be absorbed into a redefinition of $\tilde{\varphi}$ and $\tilde{H}$. We choose not to absorb dimension-dependent or L-dependent factors into our variables, as we are interested in a problem involving two dimensions at once with L the interpolating physical variable.

Similarly, $\tilde{\Gamma}$ has the asymptotic forms

$$\begin{aligned}
\tilde{\Gamma} &= \rho_d(1+\tilde{x})^{2-\alpha(d)}\,\tilde{\varphi}^{(2-\alpha(d))/\beta(d)} && \text{for} \quad z\to\infty \\
\tilde{\Gamma} &= L^{1-\alpha(d')}\rho_{d'}(1+\tilde{x})^{2-\alpha(d')}\,\tilde{\varphi}^{(2-\alpha(d'))/\beta(d')}+\frac{a_d}{2L^d} && \text{for} \quad z\to 0
\end{aligned} \tag{2.19}$$

where $\rho_d=\frac{1}{2}\alpha(d)\,\sigma_d^{\alpha(d)-1}$ and $\alpha(d)=(d-4)/(d-2)$.

For d approaching three, the film critical temperature $T_c(L)$ is driven to zero as b_d diverges with a simple pole at $d=3$. There is a similar pole in σ_{d-1}, which cancels in $\mathcal{O}(M^2, L)$, and (2.16) becomes

$$(\tau+\tilde{\varphi}^2)\,L=\frac{z}{4\pi}+\frac{1}{2\pi}\ln[1-e^{-z}] \tag{2.20}$$

in agreement with ref. 10, which restricted consideration to the zero-field case, $H=0$. It is convenient to define $w=(\tau+\tilde{\varphi}^2)\,L$, which with (2.20) implies

$$\mathcal{Q}(3, w)=\left\{2\ln\left[\frac{\exp(2\pi w)+\sqrt{\exp(4\pi w)+4}}{2}\right]\right\}^2 \tag{2.21}$$

Then (2.21) together with (2.17) specifies the universal equation of state. Similarly (2.10) with (2.11) and (2.21) specifies $\tilde{\Gamma}$.

For $L\to\infty$ we have $w\to\infty$ and we recover the three-dimensional scaling function (2.18) discussed above, and, for fixed L with $\xi_L=M^{-1}\to\infty$, the two-dimensional critical regime, which is governed by $\tau\to-\infty$, and (2.20) gives

$$\xi_L=L\exp[-2\pi(\tau+\tilde{\varphi}^2)\,L] \tag{2.22}$$

in agreement with the $H=0$ result of Singh and Pathria.[18] The limiting form of the equation of state becomes

$$\{\exp[4\pi(\tau(L)+\tilde{\varphi}^2)]\}\,\tilde{\varphi}=\tilde{H}, \qquad \text{where} \quad \tau(L)=\tau-\frac{1}{2\pi L}\ln L \tag{2.23}$$

in agreement with ref. 10.

The other special dimension of interest is $d = 4$, where $\bigcirc$ has an ultraviolet divergence. In this case it is necessary to send λ_B to infinity in such a way as to cancel the divergent contribution from $\bigcirc$ and render $\Gamma_l^{(4)}$ finite. We then find the constraint retains logarithmic corrections to scaling and becomes

$$(\tau + \bar{\varphi}^2)\, L^2 = -\frac{z^2}{(4\pi)^2} \ln\left(\frac{z^2}{z_0^2}\right) - \frac{1}{2\pi^2} \int_0^\infty dq\, \frac{q^2}{\sqrt{q^2 + z^2}} \frac{1}{\exp(\sqrt{q^2 + z^2}) - 1} \tag{2.24}$$

where $z_0 = \kappa L$ with κ a remnant microscopic scale, such that $M \ll \kappa$. Similarly the free energy scaling function is given by

$$\mathscr{G}(4, z) = -\frac{z^4}{32\pi^2}\left(\frac{1}{2} + \ln\frac{z^2}{z_0^2}\right) - \frac{1}{3\pi^2}\int_0^\infty dq\, q^2 \frac{q^2 + \frac{3}{2}z^2}{\sqrt{q^2 + z^2}[\exp(\sqrt{q^2 + z^2}) - 1]} + \frac{\pi^2}{45} \tag{2.25}$$

with $a_4 = \pi^2/45$. For $M \to 0$ with fixed L we recover the three-dimensional results above, and for $L \to \infty$ the constraint becomes

$$\frac{\tau + \bar{\varphi}^2}{\kappa^2} = -\frac{1}{(4\pi)^2} \frac{M^2}{\kappa^2} \ln\left(\frac{M^2}{\kappa^2}\right) \tag{2.26}$$

while $\tilde{\Gamma}$ becomes

$$\tilde{\Gamma} = -\frac{M^4}{64\pi^2}\left(\frac{1}{2} + \ln\frac{M^2}{\kappa^2}\right) \tag{2.27}$$

3. ENVIRONMENTALLY FRIENDLY RENORMALIZATION

The purpose of this section is to use RG techniques to recover the scaling functions in the large-N limit. As before, we assume that the finite system also èxhibits critical behavior and that $3 \leqslant d \leqslant 4$. We will restrict our considerations to a film geometry with periodic boundary conditions.

We indicate by $t_B(M)$ that temperature parameter which yields the transverse correlation length $\xi_L = M^{-1}$. The RG method is to change from the original bare parameters to new renormalized ones given by

$$t(M, \kappa) = Z_{\varphi^2}^{-1}\, t_B(M), \qquad \lambda(\kappa) = Z_\lambda(\kappa)\, \lambda_B, \qquad \bar{\varphi}(\kappa) = Z_\varphi^{-1/2} \bar{\varphi}_B$$

and

$$\Gamma^{(N,L)} = Z_{\varphi}^{N/2}(\kappa)\, Z_{\varphi^2}^{L}(\kappa)\, \Gamma_B^{(N,L)} + \delta_{N0}\delta_{Ln} A^{(n)}(\kappa), \qquad i=0, 1, 2 \quad (3.1)$$

where κ is an arbitrary renormalization scale. The renormalized vertex functions then obey the RG equation

$$\kappa \frac{d}{d\kappa} \Gamma^{(N,L)} + \left(L\gamma_{\varphi^2} - \frac{N}{2}\gamma_{\varphi} \right) \Gamma^{(N,L)} = \delta_{N0}\delta_{Ln} B^{(n)} \qquad (3.2)$$

where the Wilson functions in the above are

$$\gamma_{\varphi} = \frac{d \ln Z_{\varphi}}{d \ln \kappa} \qquad \text{and} \qquad \gamma_{\varphi^2} = -\frac{d \ln Z_{\varphi^2}}{d \ln \kappa}$$

The final Wilson function is $\gamma_{\lambda} = d \ln Z_{\lambda}/d \ln \kappa$ and is related to the beta function $\beta(\lambda)$ through the relation $\gamma_{\lambda} = \beta(\lambda)/\lambda$.

The normalization conditions which fix the particular parametrization we will use to describe physical quantities are

$$\text{(i)} \quad \Gamma_t^{(2)}|_{\mathrm{NP}} = \kappa^2; \qquad \text{(ii)} \quad \frac{\partial}{\partial k^2}\Gamma_t^{(2)}|_{\mathrm{NP}} = 1$$
$$\text{(iii)} \quad \Gamma_t^{(4)}|_{\mathrm{NP}} = \lambda; \qquad \text{(iv)} \quad \Gamma_t^{(2,1)}|_{\mathrm{NP}} = 1 \qquad (3.3)$$

where the subscript t refers to the transverse vertex functions and NP to the normalization point. A simplifying feature in considering the large-N limit is that the separate dependences on $\bar{\varphi}$ and t_B merge into a dependence on the transverse correlation length. We choose our normalization point to be at zero momentum, fixed L, and an arbitrary fiducial transverse correlation length κ^{-1}. Conditions (ii)–(iv) determine the Wilson functions, which have explicit dependence on κL, while (i) determines the relationship between κ and the physical variables t, $\bar{\varphi}$, and λ.

To generate the equation of state and free energy for comparison with the results of Section 2 we start with the differential statement

$$d\Gamma_t^{(2)} = \Gamma_t^{(2,1)}\, dt + \tfrac{1}{6}\Gamma_t^{(4)}\, d\bar{\varphi}^2 \qquad (3.4)$$

If we integrate this relation along a contour of constant magnetization $\bar{\varphi}$ from the critical isotherm $t=0$, we obtain

$$\Gamma_t^{(2)}(t, \bar{\varphi}) = \Gamma_t^{(2)}(0, \bar{\varphi}) + \int_0^t \Gamma_t^{(2,1)}(t', \bar{\varphi})\, dt' \qquad (3.5)$$

From the definitions of the Wilson functions implied by the normalization conditions (3.3), on inverting the relation (3.5), we find

$$t+\int_0^{M(0,\bar\varphi)}(2-\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x \gamma_{\varphi^2}(x')\frac{dx'}{x'}\right)\right]x\,dx$$
$$=\int_0^{M(t,\bar\varphi)}(2-\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x \gamma_{\varphi^2}(x')\frac{dx'}{x'}\right)\right]x\,dx \qquad (3.6)$$

Above we have the relation $M=M(t,\bar\varphi)$, but parametrically in terms of $M(0,\bar\varphi)$. We can determine the latter as an explicit function of $\bar\varphi$ by integrating along the critical isotherm from the critical point. If we integrate (3.4) along the critical isotherm up from the critical point, we find

$$\Gamma_t^{(2)}(0,\bar\varphi)=\int_0^{\bar\varphi^2}\frac{\Gamma_t^{(4)}(x)}{6}\,dx \qquad (3.7)$$

By choosing the normalization point NP in (3.3) on the critical isotherm such that the transverse correlation length is κ^{-1} and at zero momentum, we can express $\Gamma_t^{(4)}$ in the form

$$\Gamma_t^{(4)}=\lambda(\kappa)\exp\left[\int_\kappa^{M(0,\bar\varphi)}(\tilde\gamma_\lambda(x)-2\tilde\gamma_{\varphi^2}(x))\frac{dx}{x}\right] \qquad (3.8)$$

where $\tilde\gamma_\lambda(x)$ and $\tilde\gamma_{\varphi^2}(x)$ are the resulting Wilson functions from this prescription. Inverting (3.7), one finds

$$\bar\varphi^2=\frac{6}{\lambda}\int_0^{M(0,\bar\varphi)}(2-\tilde\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x(\tilde\gamma_\lambda(x')-\tilde\gamma_\varphi(x'))\frac{dx'}{x'}\right)\right]x\,dx \qquad (3.9)$$

The two equations (3.6) and (3.9) specify completely the relation between the transverse correlation length, temperature, and magnetization. Finally, since the transverse correlation length is infinite on the coexistence curve, i.e., $M(t_{\text{coex}},\bar\varphi)=0$, the equation of the coexistence curve is given by

$$t+\int_0^{M(0,\bar\varphi)}(2-\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x \gamma_{\varphi^2}(x')\frac{dx'}{x'}\right)\right]x\,dx=0 \qquad (3.10)$$

where $M(0,\bar\varphi)$ as a function of $\bar\varphi$ is determined by Eq. (3.9).

The specific heat and the energy density can be treated in a similar fashion to the above by beginning with the differential relation

$$d\Gamma^{(0,2)}=\Gamma^{(0,3)}\,dt+\tfrac{1}{2}\Gamma_t^{(2,2)}\,d\bar\varphi^2 \qquad (3.11)$$

By integrating along a contour of constant $\bar{\varphi}$ up from the coexistence curve, we obtain

$$\Gamma^{(0,2)} = \int_0^M (2-\gamma_\varphi(x))\left[\exp\left(2\int_\kappa^x \gamma_{\varphi^2}(x')\,\frac{dx'}{x'}\right)\right]\bar{\Gamma}^{(0,3)}(x)\,x^{d-5}\,dx \qquad (3.12)$$

and

$$\Gamma^{(0,1)} = \int_0^M (2-\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x \gamma_{\varphi^2}(x')\,\frac{dx'}{x'}\right)\right]\Gamma^{(0,2)}(x)\,x\,dx \qquad (3.13)$$

where

$$\bar{\Gamma}^{(0,3)}(M) = \frac{\Gamma^{(0,3)}(\Gamma_t^{(2)})^3}{(\Gamma_t^{(2,1)})^3\,M^d} \qquad (3.14)$$

The advantage of integrating up from the coexistence curve is that we can extract the singular part by requiring that both $\Gamma^{(0,1)}$ and $\Gamma^{(0,2)}$ vanish there. We can impose such boundary conditions only at the critical point in the special case of $\alpha < 0$, which is the case for the large-N limit. A boundary condition at some other point is also possible, but the formulas are more complicated. Finally, the free energy is given by

$$\Gamma = \Gamma|_{T_c(L)} + \int_0^M (2-\gamma_\varphi(x))\left[\exp\left(-\int_\kappa^x \gamma_{\varphi^2}(x')\,\frac{dx'}{x'}\right)\right]\Gamma^{(0,1)}x\,dx \qquad (3.15)$$

The necessary ingredients in the above prescription are γ_i, $\tilde{\gamma}_i$, and $\bar{\Gamma}^{(0,3)}$ together with the initial conditions for the integrations above. For the large-N limit we can evaluate them exactly. We find diagrammatically that

$$\bar{\Gamma}^{(0,3)} = N\,\frac{\bigcirc}{M^{d-6}} \qquad (3.16)$$

which can be related to derivatives of the scaling function $\mathscr{F}$. In terms of the floating coupling h,[2] chosen so as to make the coefficient of the quadratic term in the resulting β function unity, one finds

$$\beta(h, z) = -\varepsilon(z)\,h + h^2 \qquad (3.17)$$

and

$$\gamma_{\varphi^2}(h, z) = \gamma_\lambda(h, z) = h, \qquad \gamma_\varphi = 0 \qquad (3.18)$$

where the function $\varepsilon(z)$ is

$$\varepsilon(z) = 6z^2 \bigcirc(z)/\bigcirc(z) - 2 \tag{3.19}$$

Similarly for the critical isotherm we have

$$\tilde{\gamma}_{\varphi^2}(h, z) = \tilde{\gamma}_\lambda(h, z) = h, \qquad \tilde{\gamma}_\varphi = 0 \tag{3.20}$$

Equations (3.18) and (3.20) imply that $\tilde{\gamma}_\lambda = \gamma_{\varphi^2}$ and hence with (3.9) we find that (3.6) becomes

$$t_0 + \frac{\lambda_0 \bar{\varphi}^2}{6} = \frac{2}{L^2} \int_0^z \frac{dx}{x} \exp\left(\int_{z_0}^x (2 - h(x')) \frac{dx'}{x'} \right) \tag{3.21}$$

where h, the solution of (3.17), is

$$h(z, z_0, h_0) = \left\{ \exp\left(-\int_{z_0}^z \varepsilon(x) \frac{dx}{x} \right) \right\} \Big/ \left\{ h_0^{-1} - \int_{z_0}^z \left[\exp\left(-\int_{z_0}^x \varepsilon(x') \frac{dx'}{x'} \right) \right] \frac{dx}{x} \right\} \tag{3.22}$$

with $z_0 = \kappa_0 L$, $\lambda_0 = \lambda(\kappa_0)$, and $t_0 = t(M, \kappa_0)$. The steepest descent constraint of the large-N limit has now been recovered in the form (3.21).

In (3.22) the initial coupling is specified at the "microscopic" scale κ_0. For $d < 4$ this microscopic scale can be sent to infinity while maintaining h_0 finite. A universal floating coupling

$$h(z) = 4z^2 \bigcirc(z)/\bigcirc(z)$$

which is the separatrix solution of the differential equation, is obtained. If this solution is used in (3.21), we obtain

$$t_0 + \frac{\lambda_0 \bar{\varphi}^2}{6} = \frac{N\lambda_0}{6} M^{d-2} \mathscr{F}(d, z) \tag{3.23}$$

where

$$\lambda_0 = 6/N \bigcirc(\kappa_0)$$

is the initial dimensional coupling corresponding to the separatrix solution. In the asymptotic regime, (2.8) implies $\lambda_0 = 12\kappa_0^{4-d}/N(d-2)\,\sigma_d$. We have recovered using RG arguments the universal form of the spherical constraint (2.16), where now $\tau(L) = 6t_0/\lambda_0$. The further integration (3.15) gives the free energy scaling function. If one is interested in corrections to scaling, as is usually the case in comparing with experimental data, then κ_0

should be left finite and fitted to the data. The cases of $d=3$ and $d=4$ require special care. For $d=4$ one cannot ignore h_0, but when appropriate care is taken one recovers the results of the previous section.

4. EFFECTIVE EXPONENTS IN THE LARGE-*N* LIMIT

A very useful way of representing a large class of scaling functions is in terms of effective critical exponents, which are functions that interpolate between the constant critical exponents associated with the different asymptotic regimes that characterize the crossover.

We define, for $T > T_c(L)$ and $H=0$, an effective critical exponent

$$\nu_{\text{eff}} = -\left.\frac{d \ln \xi_L}{d \ln t_B}\right|_{H=0}$$

where ξ_L is the correlation length associated with the transverse dimensions, i.e., the correlation length in the infinite dimensions [remember that $t_B \sim T - T_c(L)$]. One finds from (2.4) that

$$\nu_{\text{eff}} = \left(\frac{1 + 6/g\mathscr{F}}{d - 2 + d \ln \mathscr{F}/d \ln z + 12/g\mathscr{F}}\right) \tag{4.1}$$

The coupling g governs the crossover from universal to mean-field-like behavior when $g \to 0$, where $\nu_{\text{eff}} \to 1/2$. In the critical regime, where $z \ll g$ and $1 \ll g$, the terms proportional to g^{-1} may be neglected, thus yielding a true universal scaling function. From (4.1) we see that as $z \to 0$, then $\nu_{\text{eff}} \to 1/(d-3)$, whereas for $z \to \infty$, $\nu_{\text{eff}} \to 1/(d-2)$. Thus ν_{eff} interpolates between the two exact asymptotic values associated with the spherical model in d and $d-1$ dimensions.

The effective exponent $\gamma_{\text{eff}} = -d \ln \chi/d \ln t_B$, where $\chi = M^{-2}$ is the susceptibility for $H=0$, is given by

$$\gamma_{\text{eff}} = 2\left(\frac{1 + 6/g\mathscr{F}}{d - 2 + d \ln \mathscr{F}/d \ln z + 12/g\mathscr{F}}\right) \tag{4.2}$$

In the mean-field limit $\gamma_{\text{eff}} \to 1$, whereas in the universal limit ($g \to \infty$) we have $\gamma_{\text{eff}} \to 2/(d-3)$ as $z \to 0$ and $\gamma_{\text{eff}} \to 2/(d-2)$ as $z \to \infty$. Thus γ_{eff} also captures both the dimensional crossover and the mean-field one.

For $T < T_c(L)$ and $H=0$, while ν_{eff} and γ_{eff} are ill defined, the effective exponent $\beta_{\text{eff}} = d \ln \bar{\varphi}/d \ln |t_B|$ is well defined. From the saddle-point equation (2.4), due to the vanishing of the transverse mass on the coexistence curve, we see from (3.23) that $\bar{\varphi}^2 = 6t/\lambda$, which implies that $\beta_{\text{eff}} = 1/2$, i.e., there is no crossover as one proceeds along the coexistence curve. This is in strong distinction to the Ising model, where there is a crossover between the critical point and the strong coupling discontinuity fixed point at $T=0$. The contrast is due to the fact that the coexistence curve is a line of first-order transitions for $N=1$ and a line of continuous transitions for $N>1$.

For the approach to the critical point as a function of field H on the critical isotherm $T=T_c(L)$ we define an effective exponent

$$\delta_{\text{eff}}=\left.\frac{d\ln H}{d\ln\bar{\varphi}}\right|_{t_B=0}$$

This implies that

$$\delta_{\text{eff}}=1+\frac{\Gamma_t^{(4)}\bar{\varphi}^2}{3\Gamma_t^{(2)}}=\left(\frac{d+2+d\ln\mathscr{F}/d\ln z+36/g\mathscr{F}}{d-2+d\ln\mathscr{F}/d\ln z+12/g\mathscr{F}}\right) \tag{4.3}$$

which interpolates between the mean-field and the respective d and d' critical exponents.

The specific heat and the energy density may also be discussed in terms of effective exponents. We may define

$$\alpha_{\text{eff}}^s=-\frac{d\ln\mathscr{C}}{d\ln t_B}\qquad\text{and}\qquad 1-\alpha_{\text{eff}}^e=\frac{d\ln\mathscr{E}}{d\ln t_B}$$

where $\mathscr{C}$ is the specific heat and $\mathscr{E}$ is the energy density for $H=0$, respectively. From the definition of α_{eff}^e we have $\alpha_{\text{eff}}^e=1-t_B\Gamma_B^{(0,2)}/\Gamma_B^{(0,1)}$, which gives

$$\alpha_{\text{eff}}^e=\left(\frac{d-4+d\ln\mathscr{F}/d\ln z}{d-2+d\ln\mathscr{F}/d\ln z+12/g\mathscr{F}}\right) \tag{4.4}$$

The specific heat effective exponent is more cumbersome; however, both vanish in the mean-field limit and in the universal limit yield effective exponents that interpolate between $\alpha(d)$ and $\alpha(d')$ as z ranges from zero to infinity. For $T<T_c$ on the coexistence curve the singular parts of the energy density and the specific heat are identically zero, implying the associated amplitudes are zero and the associated effective exponents are ill-defined.

5. EFFECTIVE EXPONENT SCALING LAWS

We now make some observations concerning certain algebraic relations between the effective exponents. Equations (4.1) and (4.2) and the fact that $\eta_{\text{eff}} \equiv 0$ due to the vanishing of γ_φ in the large-N limit yield the relation

$$\gamma_{\text{eff}} = \nu_{\text{eff}}(2 - \eta_{\text{eff}}) \tag{5.1}$$

and the expressions for γ_{eff}, α^e_{eff}, and β_{eff} imply

$$\alpha^e_{\text{eff}} + 2\beta_{\text{eff}} + \gamma_{\text{eff}} = 2 \tag{5.2}$$

Note that these relations hold even for the more general scaling functions that include the crossover to mean-field theory. The exponent α^s_{eff}, however, does not satisfy this relation. We see then that direct analogs of the normal scaling laws between exponents hold; however, here the relations are between entire scaling functions.

It is natural to ask if there are analogs of other scaling laws, in particular hyperscaling, where $2 - \alpha = \nu d$. This cannot be achieved with fixed d; however, one can define the notion of an effective dimensionality d^i_{eff} such that an effective hyperscaling law is valid. Defining $d^e_{\text{eff}} = (2 - \alpha^e_{\text{eff}})/\nu_{\text{eff}}$, one finds

$$d^e_{\text{eff}} = \left(\frac{d + d \ln \mathscr{F}/d \ln z + 24/g\mathscr{F}}{1 + 6/g\mathscr{F}} \right) \tag{5.3}$$

In the mean-field limit, $d^e_{\text{eff}} \to 4$, i.e., the upper critical dimension, as one might expect. In the limit $z \to \infty$, one finds $d^e_{\text{eff}} \to d$, while in the limit $z \to 0$ for fixed L, $d^e_{\text{eff}} \to (d-1)$. Of course one could also define an effective dimensionality via α^s_{eff} as $d^s_{\text{eff}} = (2 - \alpha^s_{\text{eff}})/\nu_{\text{eff}}$. Again it interpolates between 4, d, and $d - 1$ in the apropriate asymptotic limits. However, since the exponent α^s_{eff} did not satisfy the scaling law (5.2), we do not expect d^s_{eff} to satisfy corresponding laws and in fact it does not.

We might enquire now as to the validity of other scaling laws that involve the dimensionality explicitly. Noticing that $d^e_{\text{eff}} = 2 + 1/\nu_{\text{eff}}$ [i.e., $\nu_{\text{eff}} = 1/(d^e_{\text{eff}} - 2)$], one arrives at the scaling relations

$$\beta_{\text{eff}} = \frac{\nu_{\text{eff}}}{2}(d^e_{\text{eff}} - 2 + \eta_{\text{eff}}), \qquad \delta_{\text{eff}} = \left(\frac{d^e_{\text{eff}} + 2 - \eta_{\text{eff}}}{d^e_{\text{eff}} - 2 + \eta_{\text{eff}}} \right) \tag{5.4}$$

The standard exponent relations, including hyperscaling, tell us that two exponents and the dimensionality are sufficient to specify all other exponents. In the $N \to \infty$ limit the dimensionality alone is sufficient to determine all exponents. What we have found is that the six scaling functions α^e_{eff}, ν_{eff}, γ_{eff}, η_{eff}, δ_{eff}, and β_{eff} can analogously all be expressed in terms of the one function d^e_{eff} by natural analogs of the scaling laws.

6. CONCLUSIONS

We have studied dimensional crossover for a d-dimensional film with periodic boundary conditions in the exactly solvable large-N limit of an $O(N)$ Landau–Ginzburg–Wilson model, a model that also includes crossover to mean-field behavior. We obtained the scaling forms of the free energy and equation of state and extracted their universal limits where the crossover to mean-field theory is eliminated. We studied the model using both direct methods and the techniques of environmentally friendly renormalization and were therefore able to compare this RG approach with an exact solution obtained by other means.

In the RG approach the equation of state was found by choosing normalization conditions at a fiducial value of the transverse correlation length and integrating it along particular contours in the $(t, \bar{\varphi})$ phase diagram, first along a contour of constant magnetization and second along the critical isotherm. Both contours together yield a complete description of the phase diagram of the model.

Effective exponents were defined which exhibit both dimensional crossover and the crossover to mean-field exponents. They were found to obey natural analogs of the standard scaling laws. In the case of hyperscaling this necessitates the introduction of an effective dimensionality d_{eff} which interpolates between d and $d-1$ and in terms of which all the effective exponents can be expressed.

It is natural to ask whether or not the effective exponent laws extend to the case $N \neq \infty$. The underlying function governing all effective exponents is the free energy scaling function $\tilde{\Gamma}$, from which other thermodynamic quantities are derived by differentiation. Singular ones then have two asymptotic scaling forms, for example, a thermodynamic function P has the form

$$P \sim A_{d'}^{\pm}(L)\,|T - T_c(L)|^{-\theta_{d'}} + S_d L^{-\varphi(d)}$$

in the neighborhood[6] of $T_c(L)$, and for $L \to \infty$, $T \to T_c(\infty)$ it takes the form

$$P \sim A_d^{\pm}\,|T - T_c(\infty)|^{-\theta_d}$$

[6] Generally, for continuous transitions one would expect $S_d^+ = S_d^-$ and we assume this here.

Quite generally one can decompose the scaling function P throughout the crossover into the form

$$P = A^{\pm}(t, L)\exp\left(-\int_1^t \theta(x)\frac{dx}{x}\right) + S(t, L) \tag{6.1}$$

where at the respective asymptotic endpoints $A^{\pm}$ gives the amplitude, θ the exponent, and $S^{\pm}$ any shift that may have arisen in the thermodynamic function. However, in the crossover region this division is somewhat arbitrary. In the case of an effective exponent defined as the logarithmic derivative of $P_s = P - S_d/L^{\varphi(d)}$ with respect to $T - T_c(L)$ the decomposition is forced to take the form

$$P = A_d^{\pm}(L)\exp\left(-\int_1^t \theta_{\text{eff}}(x)\frac{dx}{x}\right) + S_d L^{-\varphi(d)} \tag{6.2}$$

In the large-N limit we have found that this decomposition has the further property that for an appropriate definition of effective exponents all the usual scaling relations are obeyed.

The reason such scaling laws are obeyed originates in the fact that $\tilde{\Gamma}$ involves t and $\tilde{\varphi}$ only in the combination

$$w = (\tilde{\varphi}^2 + \tau)\, L^{1/\nu} + b_d$$

so that derivatives with respect to τ and with respect to $\tilde{\varphi}$ are intimately related. This explains why the energy rather than the specific heat provided the effective exponent that yielded the scaling laws.

In general the decomposition (6.2) may not be possible, and is not expected to give effective exponents that obey scaling laws. However, we can choose a rather natural division into amplitude and effective exponent in the form (6.1) where the exponents do obey all the usual scaling relations, including hyperscaling. A particularly convenient choice is that associated with separatrix exponents as advocated in ref. 2, where the basic building functions are γ_{φ}, γ_{φ^2}, and γ'_{λ} evaluated on the separatrix solution of the RG flow that connects the d and d' fixed points. Under such a division the amplitudes are nonsingular functions of t and L that interpolate between the d- and d'-dimensional amplitudes and the exponents capture all the singular behavior in the scaling functions. The shift is unaffected by this choice and retains the form $S_d L^{-\varphi(d)}$.

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